

# Mathematical Proofs

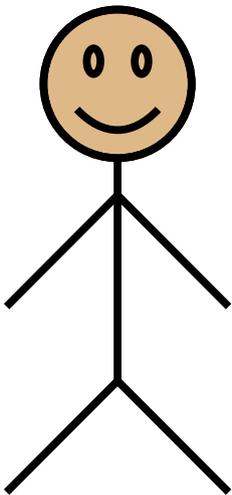
# Outline for Today

- ***How to Write a Proof***
  - Synthesizing definitions, intuitions, and conventions.
- ***Proofs on Numbers***
  - Working with odd and even numbers.
- ***Universal and Existential Statements***
  - Two important classes of statements.
- ***Variable Ownership***
  - Who owns what?

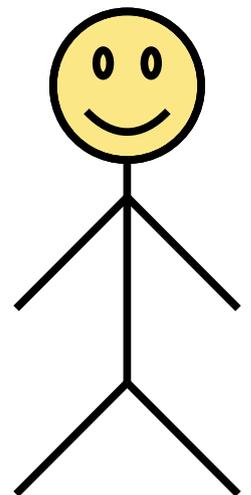
What is a Proof?

# Proof as Dialog

- A mathematical proof is a dialog between two parties: a ***proof writer*** and a ***proof reader***.
  - The ***proof writer*** knows a mathematical fact.
  - The ***proof reader*** is honest but skeptical.
- The proof writer's job is to take the reader on a journey from ignorance to understanding.



***Proof Writer (You)***



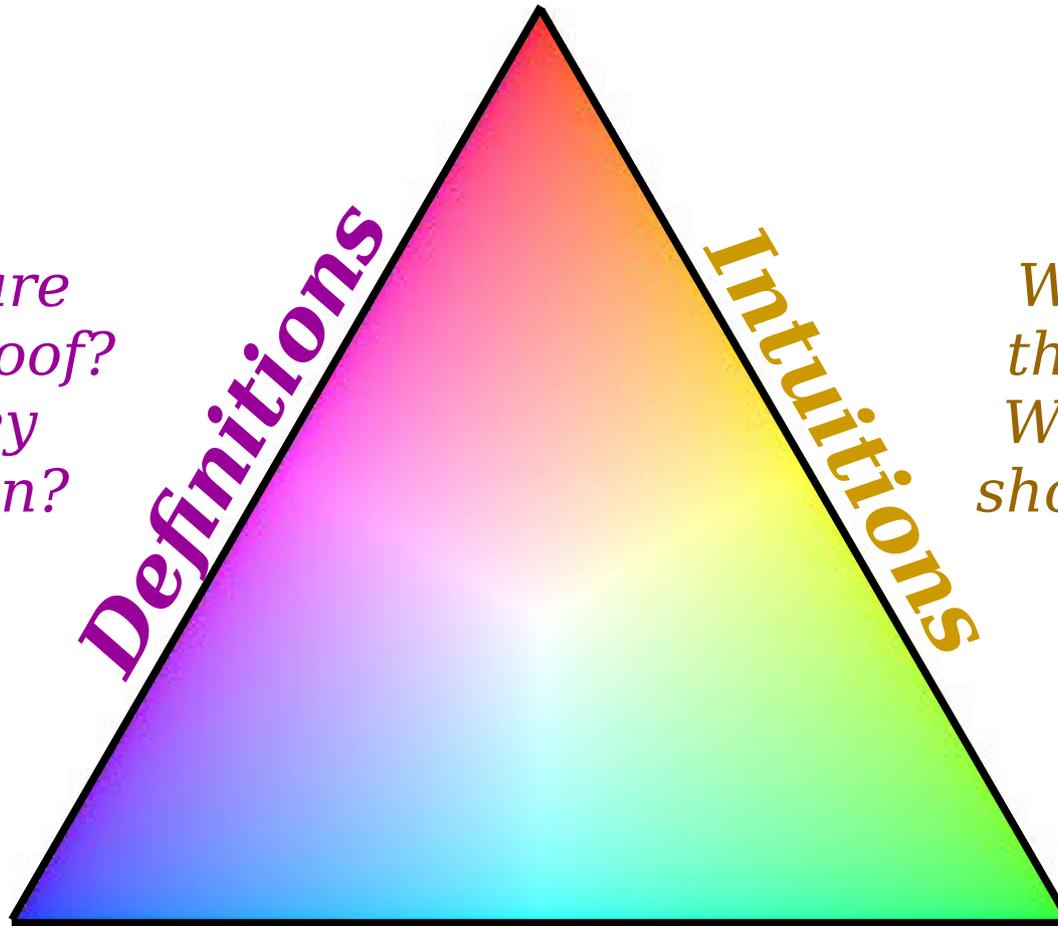
***Proof Reader***

*What terms are  
used in this proof?  
What do they  
formally mean?*

***Definitions***

***Intuitions***

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*



***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

# Writing our First Proof

***Theorem:*** If  $n$  is an even integer,  
then  $n^2$  is even.

*What terms are  
used in this proof?  
What do they  
formally mean?*

***Definitions***

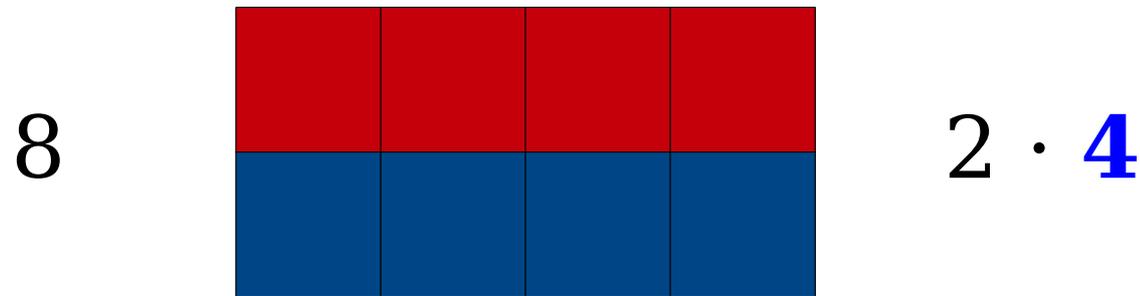
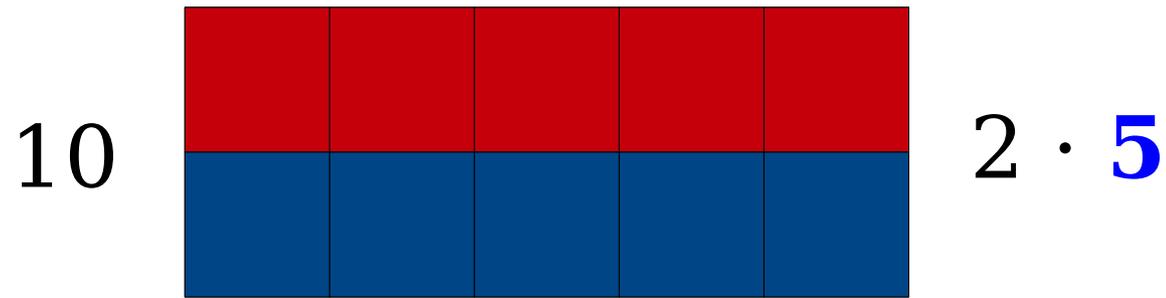
***Intuitions***

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*

***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

***Theorem:*** If  $n$  is an **even** integer,  
then  $n^2$  is **even**.



---

An integer  $n$  is called ***even*** if there is an integer  $k$  where  $n = 2k$ .

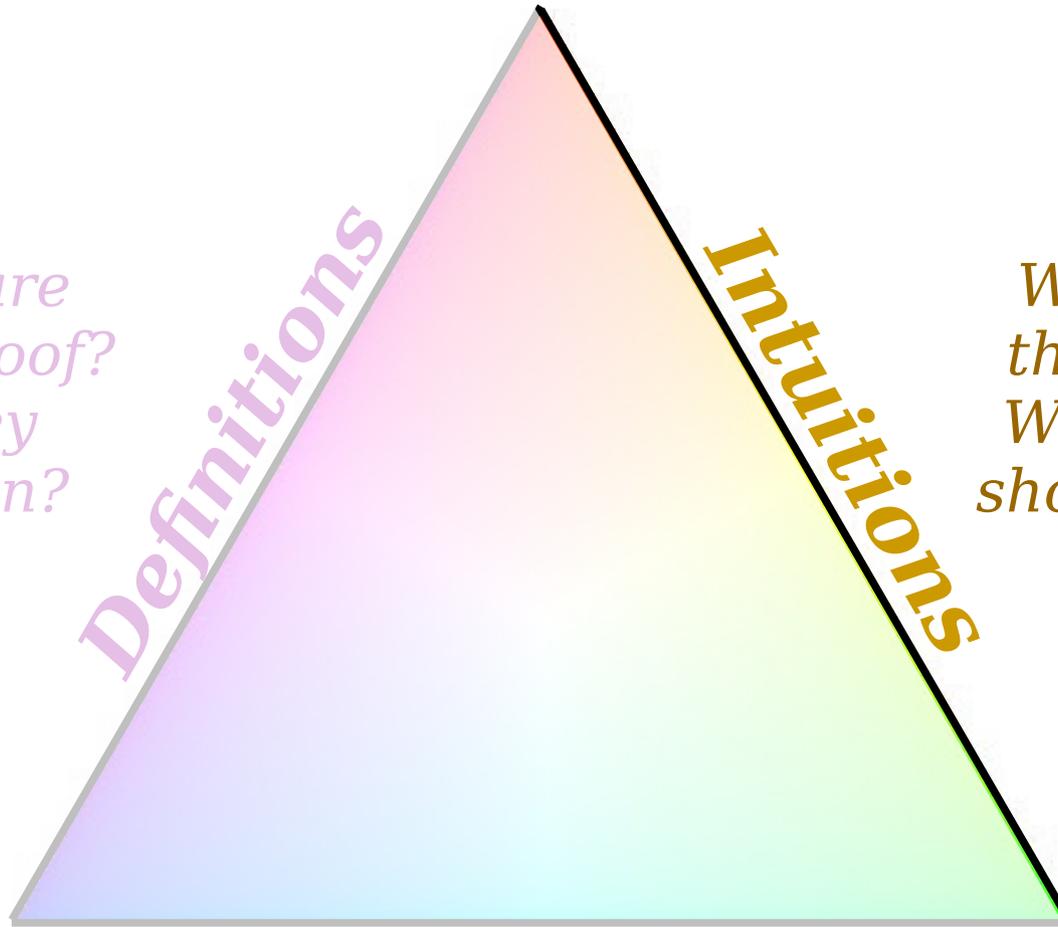
***Theorem:*** If  $n$  is an even integer,  
then  $n^2$  is even.

*What terms are used in this proof?  
What do they formally mean?*

***Definitions***

*What does this theorem mean?  
Why, intuitively, should it be true?*

***Intuitions***



***Conventions***

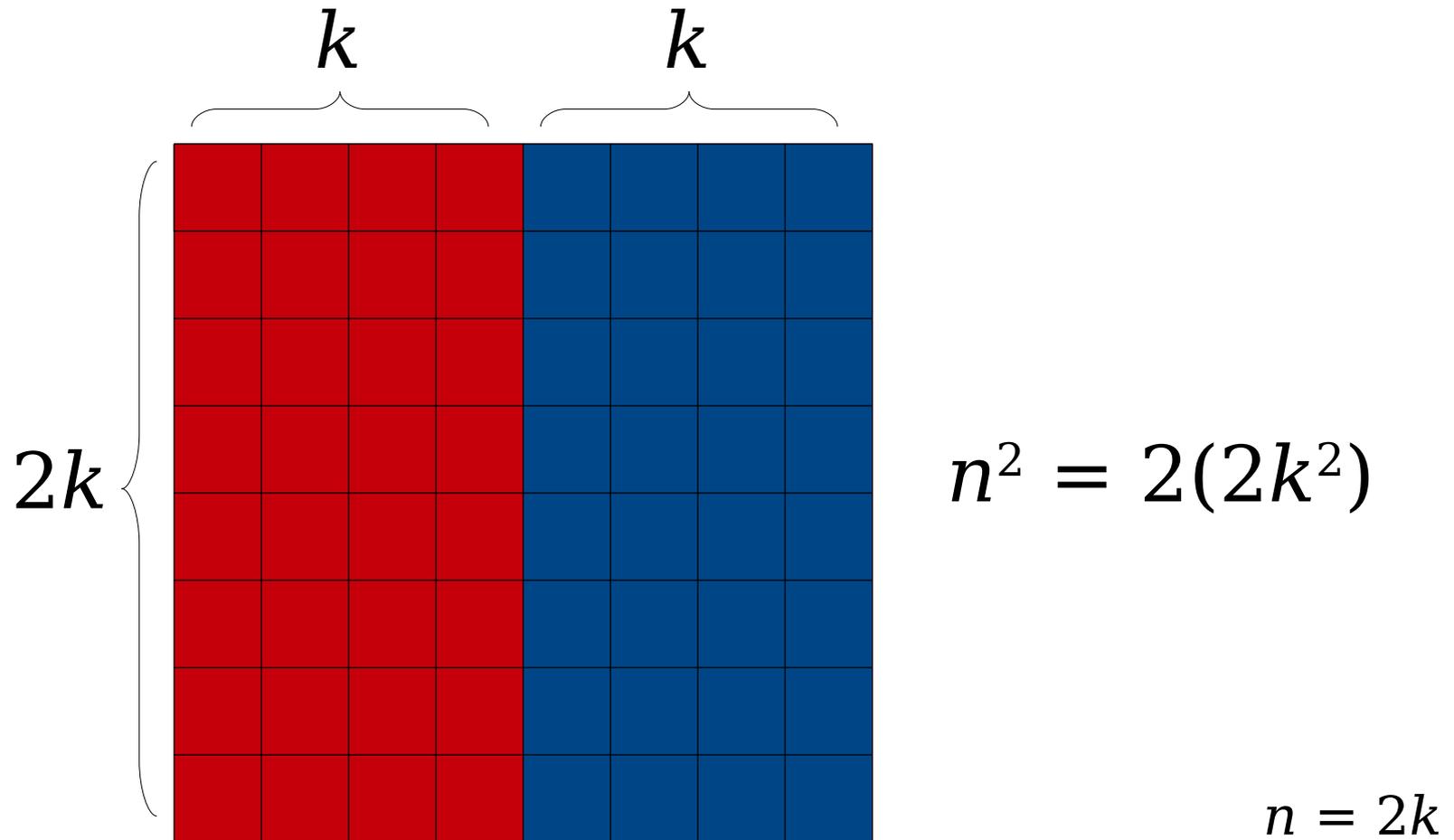
*What is the standard format for writing a proof?  
What are the techniques for doing so?*

# Let's Try Some Examples!

---

***Theorem:*** If  $n$  is an even integer, then  $n^2$  is even.

# Let's Draw Some Pictures!



**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

*What terms are  
used in this proof?  
What do they  
formally mean?*

*Definitions*

*Intuitions*

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*

***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

# Our First Proof!

**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

**Proof:** Assume  $n$  is an even integer. We want to show that  $n^2$  is even.

Since  $n$  is even, there is some integer  $k$  such that  $n = 2k$ . This means that

$$\begin{aligned}n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2).\end{aligned}$$

From this, we see that there is an integer  $m$  (namely,  $2k^2$ ) where  $n^2 = 2m$ . Therefore,  $n^2$  is even, which is what we wanted to show. ■

# Our First Proof!

**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

**Proof:** Assume  $n$  is an even integer. We want to show that  $n^2$  is even.

Since  $n$  is even, there is some integer  $k$  such that  $n = 2k$ . This means that

$$\begin{aligned}n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2).\end{aligned}$$

This symbol  
means "end of  
proof"



From this, we see that there is an integer  $m$  (namely,  $2k^2$ ) where  $n^2 = 2m$ . Therefore,  $n^2$  is even, which is what we wanted to show. ■

# Our First Proof!

**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

**Proof:** Assume  $n$  is an even integer. We want to show that  $n^2$  is even.

Since  $n$  is even, there is some integer  $k$  such that  $n = 2k$ . This means that

To prove a statement of the form

**“If  $P$  is true, then  $Q$  is true,”**

start by asking the reader to assume that  **$P$**  is true.

From this, we see that  $n^2 = (2k)^2 = 4k^2$  (namely,  $2k^2$ ) which is even, which is

# Our First Proof!

**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

**Proof:** Assume  $n$  is an even integer. We want to show that  $n^2$  is even.

Since  $n$  is even, there is some integer  $k$  such that  $n = 2k$ . This means that

To prove a statement of the form

**“If  $P$  is true, then  $Q$  is true,”**

From this, we assume  $P$  is true, then need to show (namely,  $2k^2$  is even, which is true) that  $Q$  is true. Here, we're telling the reader where we're headed.

# Our First Proof!

**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

**Proof:** Assume  $n$  is an even integer. We want to show that  $n^2$  is even.

Since  $n$  is even, there is some integer  $k$  such that  $n = 2k$ . This means that

This is the definition of an even integer. We need to use this definition to make this proof rigorous.

From this, we can see that  $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$  (namely,  $2k^2$ ) where  $n^2 = 2m$ . Therefore,  $n^2$  is even, which is what we wanted to show. ■

# Our First Proof!

**Theorem:** If

**Proof:** Assume  
show that

Since  $n$  is even,

that  $n = 2k$ . This means that

$$\begin{aligned}n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2).\end{aligned}$$

From this, we see that there is an integer  $m$  (namely,  $2k^2$ ) where  $n^2 = 2m$ . Therefore,  $n^2$  is even, which is what we wanted to show. ■

Notice how we use the value of  $k$  that we obtained above. Giving names to quantities, allows us to manipulate them. This is similar to variables in programs.

# Our First Proof!

**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

**Proof:** Assume  $n$  is an even integer. We want to show that  $n^2$  is even.

Since  
that  $n$

Our ultimate goal is to prove that  $n^2$  is even. This means that we need to find some  $m$  where

$n^2 = 2m$ . Here, we're explicitly showing how we can do that.

From this, we see that there is an integer  $m$  (namely,  $2k^2$ ) where  $n^2 = 2m$ . Therefore,  $n^2$  is even, which is what we wanted to show. ■

# Our First Proof!

**Theorem:** If  $n$  is an even integer, then  $n^2$  is even.

**Proof:** Assume  $n$  is an even integer. We want to show that  $n^2$  is even.

Since  $n$  is even, there is some integer  $k$  such that  $n = 2k$ . This means that

$$\begin{aligned}n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2)\end{aligned}$$

Hey, that's what we said we were going to do! We're done.

From this, we see that there is an integer  $m$  (namely,  $2k^2$ ) where  $n^2 = 2m$ . Therefore,  $n^2$  is even, which is what we wanted to show. ■

Our Next Proof

***Theorem:*** For any integers  $m$  and  $n$ ,  
if  $m$  and  $n$  are odd, then  $m + n$  is even.

*What terms are  
used in this proof?  
What do they  
formally mean?*

***Definitions***

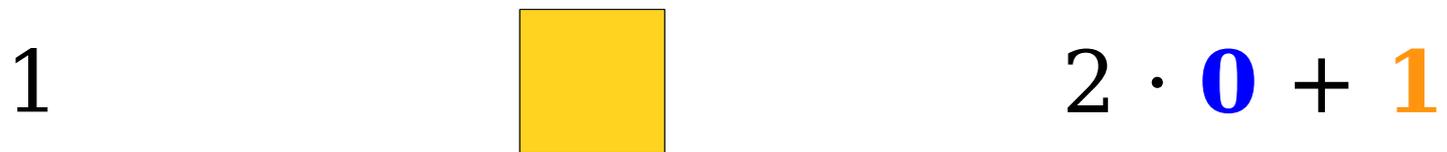
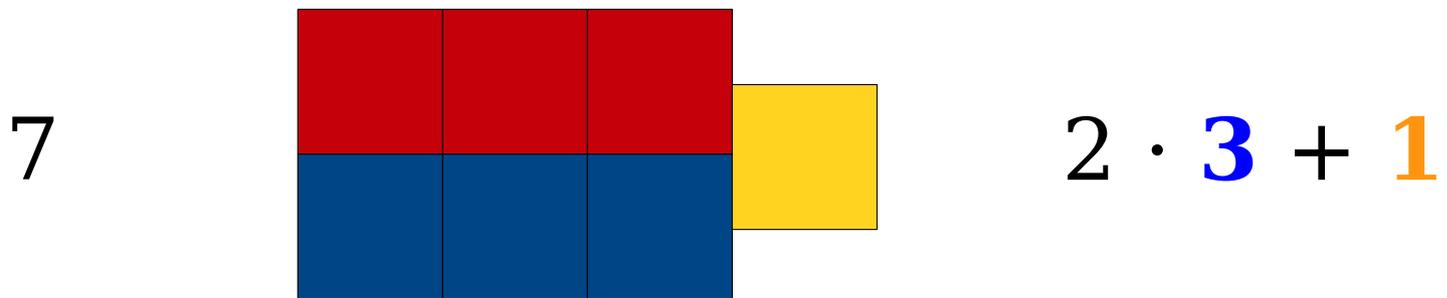
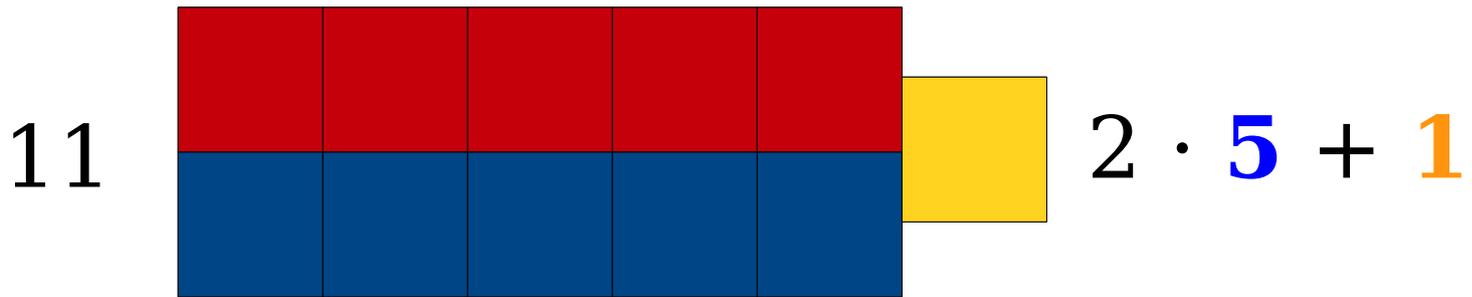
***Intuitions***

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*

***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

***Theorem:*** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are **odd**, then  $m + n$  is even.



---

An integer  $n$  is called **odd** if there is an integer  $k$  where  $n = 2k + 1$ .

Going forward, we'll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

***Theorem:*** For any integers  $m$  and  $n$ ,  
if  $m$  and  $n$  are odd, then  $m + n$  is even.

*What terms are  
used in this proof?  
What do they  
formally mean?*

***Definitions***

***Intuitions***

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*

***Conventions***

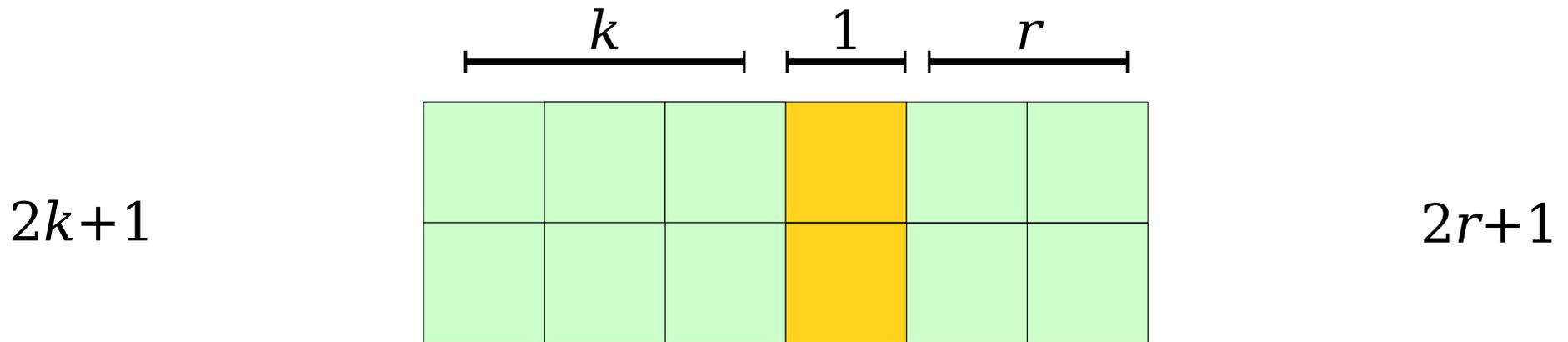
*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

# Let's Try Some Examples!

---

***Theorem:*** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are odd, then  $m+n$  is even.

# Let's Do Some Math!



$$(2k+1) + (2r+1) = 2(k + r + 1)$$

---

**Theorem:** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are odd, then  $m+n$  is even.

*What terms are  
used in this proof?  
What do they  
formally mean?*

*Definitions*

*Intuitions*

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*

***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

**Theorem:** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are odd, then  $m + n$  is even.

**Proof:** Consider any arbitrary integers  $m$  and  $n$  where  $m$  and  $n$  are odd. We need to show that  $m + n$  is even.

Since  $m$  is odd, we know that there is an integer  $k$  where

$$m = 2k + 1. \quad (1)$$

Similarly, because  $n$  is odd there must be some integer  $r$  such that

$$n = 2r + 1. \quad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \quad (3)$$

Equation (3) tells us that there is an integer  $s$  (namely,  $k + r + 1$ ) such that  $m + n = 2s$ . Therefore, we see that  $m + n$  is even, as required. ■

**Theorem:** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are odd, then  $m + n$  is even.

**Proof:** Consider any arbitrary integers  $m$  and  $n$  where  $m$  and  $n$  are odd. We need to show that  $m + n$  is even.

Since  $m$  is odd,

Similarly, because

By adding equation

Equation (3) tells us that  $m + n$  is even, as required. ■

We ask the reader to make an *arbitrary choice*. Rather than specifying what  $m$  and  $n$  are, we're signaling to the reader that they could, in principle, supply any choices of  $m$  and  $n$  that they'd like.

By letting the reader pick  $m$  and  $n$  arbitrarily, anything we prove about  $m$  and  $n$  will generalize to all possible choices for those values.

**Theorem:** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are odd, then  $m + n$  is even.

**Proof:** Consider any arbitrary integers  $m$  and  $n$  where  $m$  and  $n$  are odd. We need to show that  $m + n$  is even.

Since  $m$  is

To prove a statement of the form

**“If  $P$  is true, then  $Q$  is true,”**

Similarly, b

n that

By adding

start by asking the reader to assume that  **$P$**  is true.

$$= 2k + 2r + 2$$

$$= 2(k + r + 1). \quad (3)$$

Equation (3) tells us that there is an integer  $s$  (namely,  $k + r + 1$ ) such that  $m + n = 2s$ . Therefore, we see that  $m + n$  is even, as required. ■

**Theorem:** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are odd, then  $m + n$  is even.

**Proof:** Consider any arbitrary integers  $m$  and  $n$  where  $m$  and  $n$  are odd. We need to show that  $m + n$  is even.

Since  $m$  is odd, we know that there is an integer  $k$  such that

To prove a statement of the form

**“If  $P$  is true, then  $Q$  is true,”**

Similarly,

such that

By adding

after assuming  $P$  is true, you need to show that  $Q$  is true.

$$= 2k + 2r + 2$$

$$= 2(k + r + 1). \quad (3)$$

Equation (3) tells us that there is an integer  $s$  (namely,  $k + r + 1$ ) such that  $m + n = 2s$ . Therefore, we see that  $m + n$  is even, as required. ■

**Theorem:** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are odd, then  $m + n$  is even.

**Proof:** Consider any odd. We need to show that  $m + n$  is even. Since  $m$  is odd, we can write

Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

$$m = 2k + 1. \quad (1)$$

Similarly, because  $n$  is odd there must be some integer  $r$  such that

$$n = 2r + 1. \quad (2)$$

By adding equations (1) and (2) we learn that

$$\begin{aligned} m + n &= 2k + 1 + 2r + 1 \\ &= 2k + 2r + 2 \\ &= 2(k + r + 1). \end{aligned} \quad (3)$$

Equation (3) tells us that there is an integer  $s$  (namely,  $k + r + 1$ ) such that  $m + n = 2s$ . Therefore, we see that  $m + n$  is even, as required. ■

**Theorem:** For any integers  $m$  and  $n$ , if  $m$  and  $n$  are odd, then  $m + n$  is even.

**Proof:** Consider any arbitrary integers  $m$  and  $n$  where  $m$  and  $n$  are odd. We need to show that  $m + n$  is even.

Since  $m$  is odd, we know that there is an integer  $k$  where

$$m = 2k + 1. \quad (1)$$

Similarly, because  $n$  is odd there must be some integer  $r$  such that

$$(2)$$

This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test - if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

learn that

$$+ 2r + 1$$

$$+ 2$$

$$+ 1).$$

$$(3)$$

integer  $s$  (namely,  $k + r + 1$ )

see that  $m + n$  is even, as

# Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - **Theorem:** The sum and difference of any two even numbers is even.
  - **Theorem:** The sum and difference of an odd number and an even number is odd.
  - **Theorem:** The product of any integer and an even number is even.
  - **Theorem:** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

# Universal and Existential Statements

***Theorem:*** For any odd integer  $n$ ,  
there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

*What terms are  
used in this proof?  
What do they  
formally mean?*

***Definitions***

***Intuitions***

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*

***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

***Theorem:*** For any odd integer  $n$ ,  
there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

This result is true for every possible choice of odd integer  $n$ . It'll work for  $n = 1$ ,  $n = 137$ ,  $n = 103$ , etc.

***Theorem:*** For any odd integer  $n$ ,  
there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

We aren't saying this is true for every choice of  $r$  and  $s$ . Rather, we're saying that *somewhere out there* are choices of  $r$  and  $s$  where this works.

# Universal vs. Existential Statements

- A ***universally-quantified statement*** is a statement of the form

**For all  $x$ , [some-property] holds for  $x$ .**

- We've seen how to prove these statements.
- An ***existentially-quantified statement*** is a statement of the form

**There is some  $x$  where [some-property] holds for  $x$ .**

- How do you prove an existentially-quantified statement?

# Proving an Existential Statement

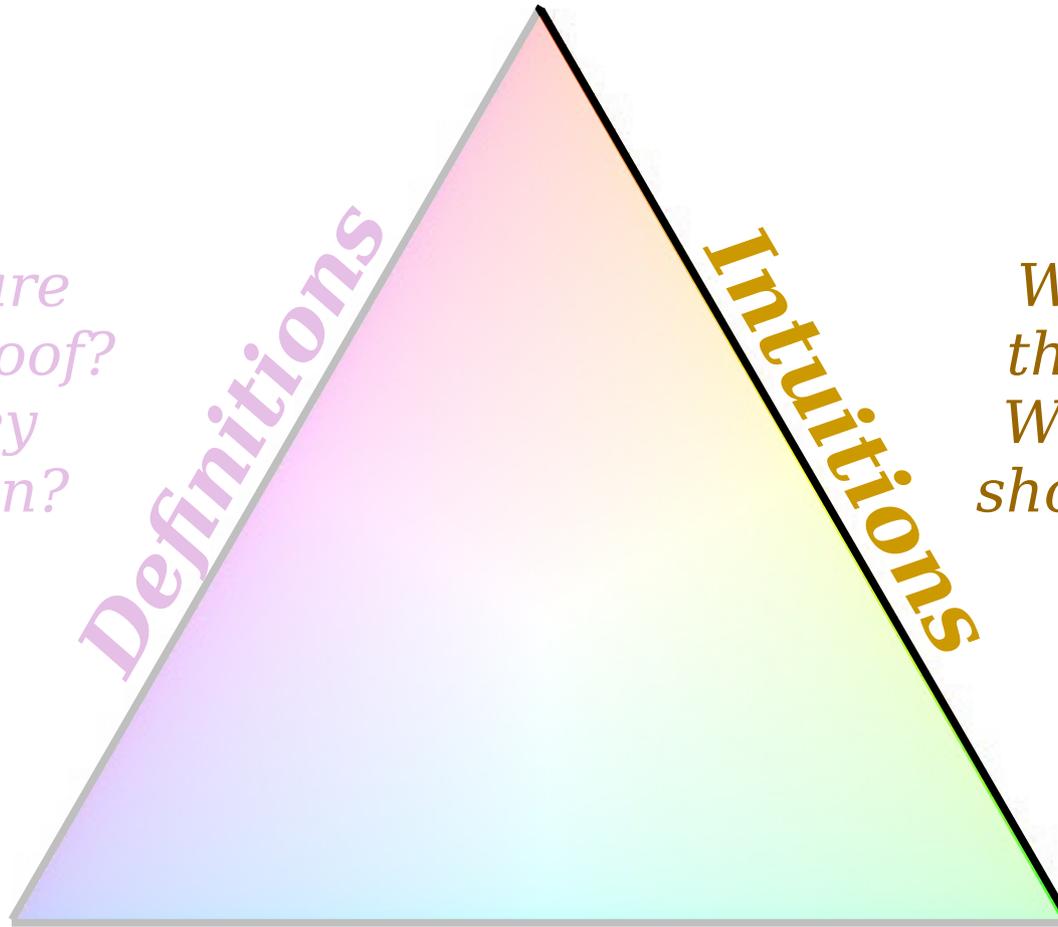
- Over the course of the quarter, we will see several different ways to prove an existentially-quantified statement of the form  
**There is an  $x$  where [some-property] holds for  $x$ .**
- ***Simplest approach:*** Search far and wide, find an  $x$  that has the right property, then show why your choice is correct.

*What terms are used in this proof?  
What do they formally mean?*

**Definitions**

*What does this theorem mean?  
Why, intuitively, should it be true?*

**Intuitions**



**Conventions**

*What is the standard format for writing a proof?  
What are the techniques for doing so?*

# Let's Try Some Examples!

$$1 = 2 \cdot \mathbf{0} + 1 = \mathbf{1}^2 - \mathbf{0}^2$$

$$3 = 2 \cdot \mathbf{1} + 1 = \mathbf{2}^2 - \mathbf{1}^2$$

$$5 = 2 \cdot \mathbf{2} + 1 = \mathbf{3}^2 - \mathbf{2}^2$$

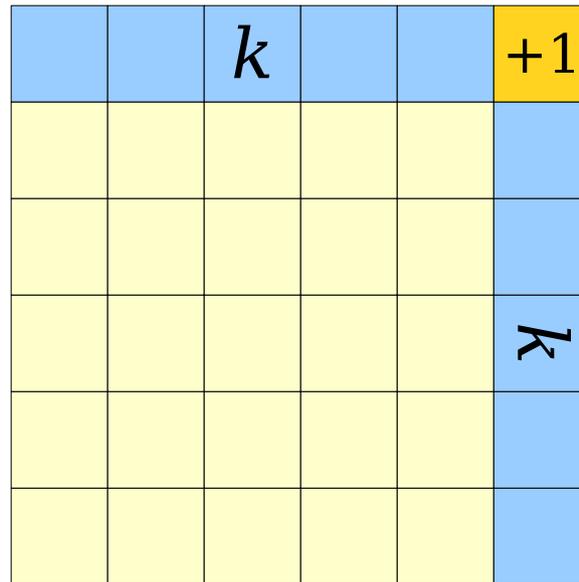
$$7 = 2 \cdot \mathbf{3} + 1 = \mathbf{4}^2 - \mathbf{3}^2$$

$$9 = 2 \cdot \mathbf{4} + 1 = \mathbf{5}^2 - \mathbf{4}^2$$

---

***Theorem:*** For any odd integer  $n$ ,  
there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

# Let's Draw Some Pictures!



$$(k+1)^2 - k^2 = 2k+1$$

---

***Theorem:*** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

*What terms are  
used in this proof?  
What do they  
formally mean?*

*Definitions*

*Intuitions*

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*

***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

**Theorem:** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Proof:** Let  $n$  be an arbitrary odd integer. We will show that there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

Since  $n$  is odd, we know there is an integer  $k$  where  $n = 2k + 1$ . Now, let  $r = k+1$  and  $s = k$ . Then we see that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

This means that  $r^2 - s^2 = n$ , which is what we needed to show. ■

**Theorem:** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Proof:** Let  $n$  be an arbitrary odd integer. We will show that there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

Since  $n$  is odd,  
 $n = 2k + 1$   
that

We ask the reader to make an *arbitrary choice*. Rather than specifying what  $n$  is, we're signaling to the reader that they could, in principle, supply any choice  $n$  that they'd like.

there  
we see

$$= 2k + 1$$

$$= n.$$

This means that  $r^2 - s^2 = n$ , which is what we needed to show. ■

**Theorem:** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Proof:** Let  $n$  be an arbitrary odd integer. We will show that there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

Since  $n$  is odd, we know that  $n = 2k + 1$ . Now, let  $r = k + 1$  and  $s = k$ . Then we have that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

As always, it's helpful to write out what we need to demonstrate with the rest of the proof.

This means that  $r^2 - s^2 = n$ , which is what we needed to show. ■

**Theorem:** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Proof:** Let  $n$  be an arbitrary odd integer. We will show that there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

Since  $n$  is odd, we know there is an integer  $k$  where  $n = 2k + 1$ . Now, let  $r = k+1$  and  $s = k$ . Then we see that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 \\ &= k^2 + 2k + 1 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

This means that  $r^2 - s^2 = n$ , which is what we wanted to show. ■

We're trying to prove an existential statement. The easiest way to do that is to just give concrete choices of the objects being sought out.

**Theorem:** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Proof:** Let  $n$  be an arbitrary odd integer. We will show that there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

Since  $n$  is odd, we know there is an integer  $k$  where  $n = 2k + 1$ . Now, let  $r = k+1$  and  $s = k$ . Then we see that

$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \\ &= 2k + 1 \\ &= n. \end{aligned}$$

This means that  $r^2 - s^2 = n$ , which is what we needed to show. ■

Who Owns What?

**Theorem:** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Proof:** Let  $n$  be an arbitrary odd integer. We will show that there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

Since  $n$  is odd, we can write  $n = 2k + 1$ . Now, we want to show that

$$r^2 - s^2 = n$$

To prove a universal statement about a variable  $n$ , ask the person reading the proof to make the choice for  $n$ . You (the proof writer) don't get to pick its value.

where we see

$$= n.$$

This means that  $r^2 - s^2 = n$ , which is what we needed to show. ■

**Theorem:** For any odd integer  $n$ , there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

**Proof:** Let  $n$  be an arbitrary odd integer. We will show that there exist integers  $r$  and  $s$  where  $r^2 - s^2 = n$ .

Since  $n$  is odd, we know there is an integer  $k$  where  $n = 2k + 1$ . Now, let  $r = k+1$  and  $s = k$ . Then we see that

$$r^2 - s^2 = (k+1)^2 - k^2$$

To prove an existential statement about a variable  $r$ , you (the proof writer) should give an explicit choice for  $r$ . Don't ask the reader to pick the value.

This means that  $r^2 - s^2 = n$ , which is what we needed to show. ■

Check the appendix to this slide deck for more about who gets to choose values.

**Time-Out for Announcements!**

# Working in Pairs

- Problem Set Zero is due this Friday at 2:30PM. It must be completed individually.
- After that, the remaining problem sets can be done individually or in pairs.
- We have advice about how to work effectively in pairs up on the course website - check the "Guide to Partners."
- Want to work in a pair, but don't know who to work with? Fill out [\*\*\*this Google form\*\*\*](#) and we'll connect you with a partner on Friday.

# CURIS Poster Session

- CURIS is the CS department's undergraduate research program. It's a great way to get involved in research!
- There's a CURIS poster session showcasing work from the summer going on from 3PM - 5PM Friday in the Engineering Quad. Feel free to stop on by!
- Interested in seeing what research projects are open right now? Visit <https://curis.stanford.edu>.
- Have questions about research or how CURIS works? Email the CURIS mentors, PhD students who answer questions about research:

**[curis-mentors@cs.stanford.edu](mailto:curis-mentors@cs.stanford.edu)**

# Qt Creator Help Session

- The lovely CS106B staff have invited all y'all to join them for a Qt Creator Help Session this evening if you're having trouble getting Qt Creator up and running on your system.
- Runs **7:00PM - 9:00PM** on the third floor of the Durand Building.
- SCPD students - please reach out to us if you need help setting things up. We'll do our best to help out.

Back to CS103!

***Theorem:*** If  $n$  is an integer,  
then  $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$ .

*What terms are  
used in this proof?  
What do they  
formally mean?*

***Definitions***

***Intuitions***

*What does this  
theorem mean?  
Why, intuitively,  
should it be true?*

***Conventions***

*What is the standard  
format for writing a proof?  
What are the techniques  
for doing so?*

# Floors and Ceilings

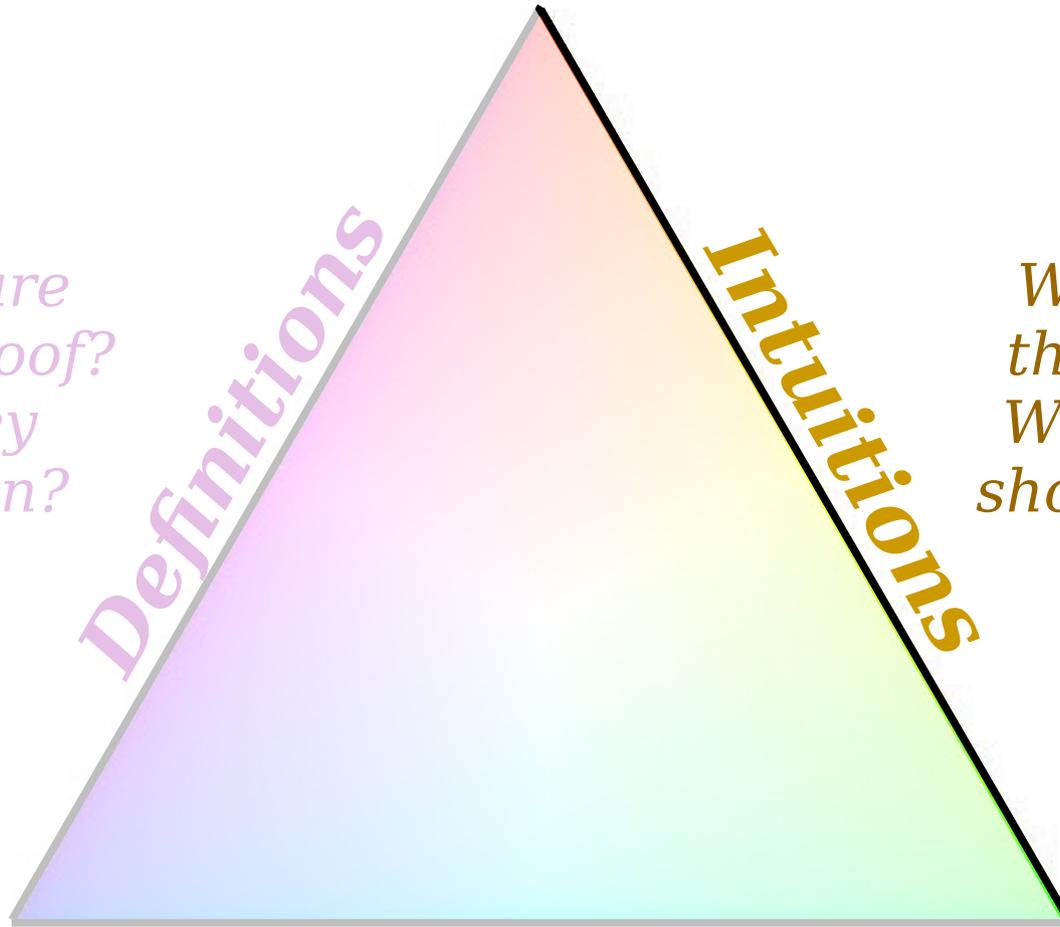
- The notation  $\lceil x \rceil$  represents the **ceiling** of  $x$ , the smallest integer greater than or equal to  $x$ .
  - What is  $\lceil 1 \rceil$ ? What's  $\lceil 1.2 \rceil$ ? What's  $\lceil -1.2 \rceil$ ?
  - **Intuition:** Start at  $x$  on the number line, then move to the right until you hit a tick mark.
- The notation  $\lfloor x \rfloor$  represents is the **floor** of  $x$ , the largest integer less than or equal to  $x$ .
  - What is  $\lfloor 1 \rfloor$ ? What's  $\lfloor 1.2 \rfloor$ ? What's  $\lfloor -1.2 \rfloor$ ?
  - **Intuition:** Start at  $x$  on the number line, then move to the left until you hit a tick mark.

*What terms are used in this proof?  
What do they formally mean?*

**Definitions**

*What does this theorem mean?  
Why, intuitively, should it be true?*

**Intuitions**



**Conventions**

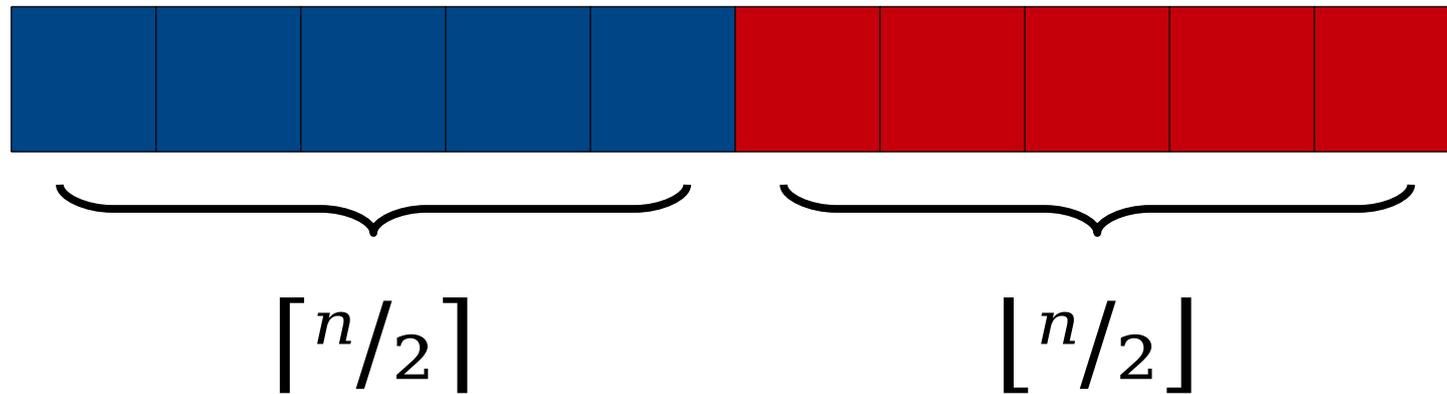
*What is the standard format for writing a proof?  
What are the techniques for doing so?*

# Let's Try Some Examples!

---

**Theorem:** If  $n$  is an integer, then  $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$ .

# Let's Draw Some Pictures!

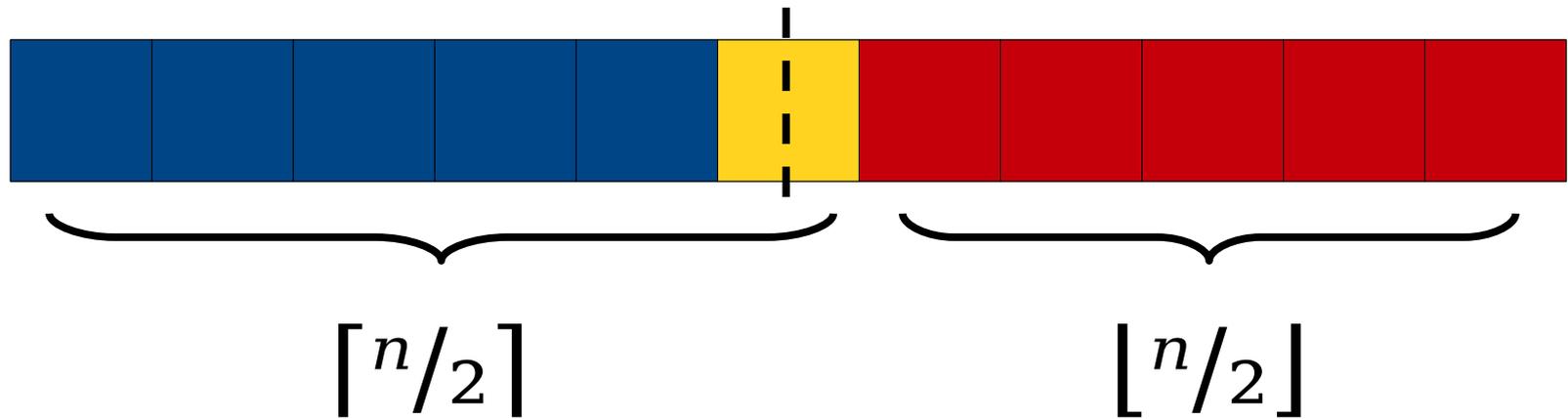


$$n = 2k$$

---

**Theorem:** If  $n$  is an integer, then  $\lfloor n/2 \rfloor + \lfloor n/2 \rfloor = n$ .

# Let's Draw Some Pictures!



$$n = 2k + 1$$

---

**Theorem:** If  $n$  is an integer, then  $\lfloor n/2 \rfloor + \lfloor n/2 \rfloor = n$ .

*What terms are used in this proof?  
What do they formally mean?*

*Definitions*

*Intuitions*

*What does this theorem mean?  
Why, intuitively, should it be true?*

***Conventions***

*What is the standard format for writing a proof?  
What are the techniques for doing so?*

**Theorem:** If  $n$  is an integer, then  $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ .

**Proof:** Let  $n$  be an integer. We will show that  $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ . To do so, we consider two cases:

*Case 1:*  $n$  is even. This means there is an integer  $k$  such that  $n = 2k$ . Some algebra then tells us that

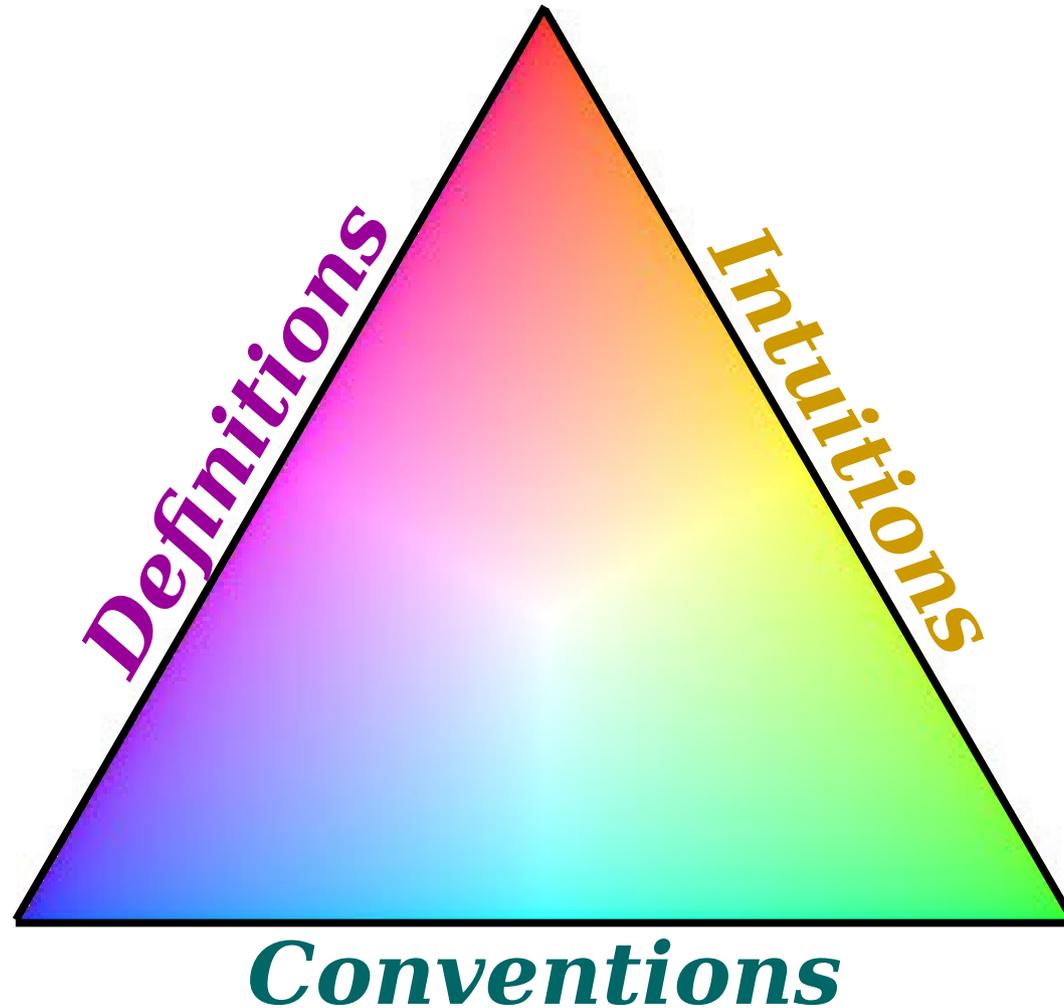
$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil &= \left\lfloor \frac{2k}{2} \right\rfloor + \left\lceil \frac{2k}{2} \right\rceil \\ &= \lfloor k \rfloor + \lceil k \rceil \\ &= 2k \\ &= n.\end{aligned}$$

*Case 2:*  $n$  is odd. Then there's an integer  $k$  where  $n = 2k + 1$ , and

$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil &= \left\lfloor \frac{2k+1}{2} \right\rfloor + \left\lceil \frac{2k+1}{2} \right\rceil \\ &= \left\lfloor k + \frac{1}{2} \right\rfloor + \left\lceil k + \frac{1}{2} \right\rceil \\ &= (k+1) + k \\ &= 2k+1 \\ &= n.\end{aligned}$$

In either case, we see that  $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ , as required. ■

To Recap



---

Writing a good proof requires a blend of definitions, intuitions, and conventions.

An integer  $n$  is **even** if there is an integer  $k$  where  $n = 2k$ .

An integer  $n$  is **odd** if there is an integer  $k$  where  $n = 2k+1$ .

---

Definitions tell us what we need to do in a proof. Many proofs directly reference these definitions.

**Let's Draw Some Pictures!**

**Let's Do Some Math!**

**Let's Try Some Examples!**

---

Building intuition for results requires creativity,  
trial, and error.

- Prove universal statements by making arbitrary choices.
- Prove existential statements by making concrete choices.
- Prove “If  $P$ , then  $Q$ ” by assuming  $P$  and proving  $Q$ .
- Write in complete sentences.
- Number sub-formulas when referring to them.
- Summarize what was shown in proofs by cases.
- Articulate your start and end points.

---

Mathematical proofs have established conventions that increase rigor and readability.

# Your Action Items

- ***Read “Guide to  $\in$  and  $\subseteq$ .”***
  - You’ll want to have a handle on how these concepts are related, and on how they differ.
- ***Read “Guide to Proofs.”***
  - This resource covers proofwriting strategies and conventions and is an essential complement to this lecture.
- ***Read “Guide to Partners.”***
  - It’s all about how to work effectively in pairs. Mull this over so you’re ready to go for Problem Set 1.
- ***Finish and submit Problem Set 0.***
  - Don’t put this off until the last minute!

# Next Time

- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does “if  $P$ , then  $Q$ ” mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

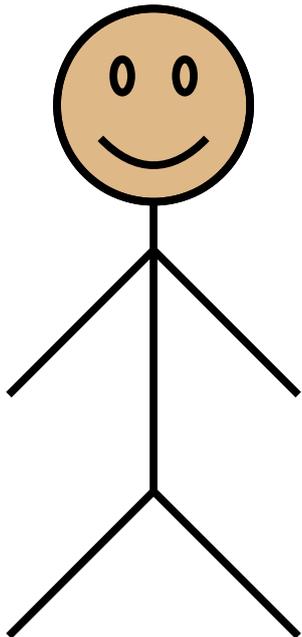
## Appendix: *Proofs as Dialogs*

# Proofs as a Dialog

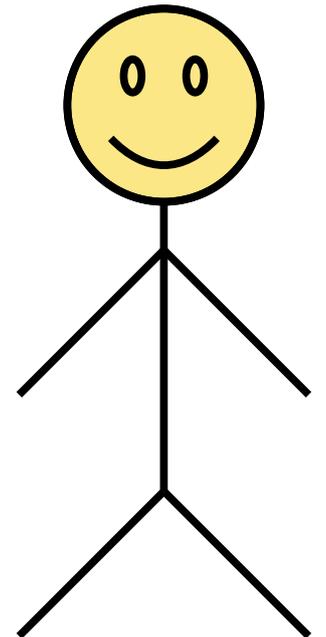
Let  $n$  be an arbitrary odd integer.

Since  $n$  is an odd integer, there is an integer  $k$  such that  $n = 2k + 1$ .

Now, let  $z = k - 34$ .



*Proof Writer (You)*



*Proof Reader*

# Proofs as a Dialog

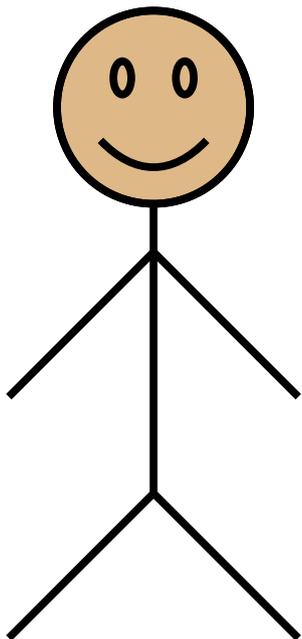
Let  $n$  be an arbitrary odd integer.

Since  $n$  is an odd integer, there is an integer  $k$  such that  $n = 2k + 1$ .

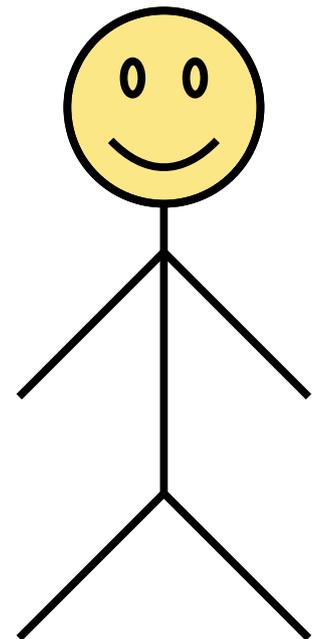
Now, let  $z = k - 34$ .

$$n = 137$$

*Reader Picks*



*Proof Writer (You)*



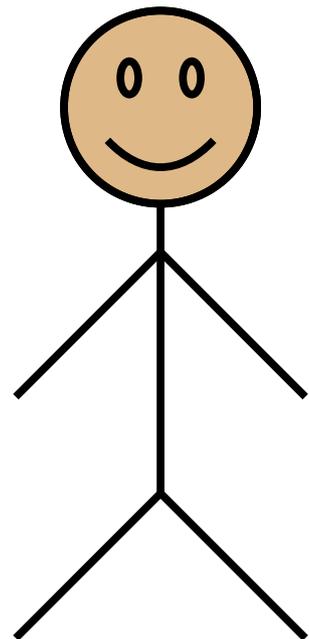
*Proof Reader*

# Proofs as a Dialog

Let  $n$  be an arbitrary odd integer.

Since  $n$  is an odd integer, there is an integer  $k$  such that  $n = 2k + 1$ .

Now, let  $z = k - 34$ .



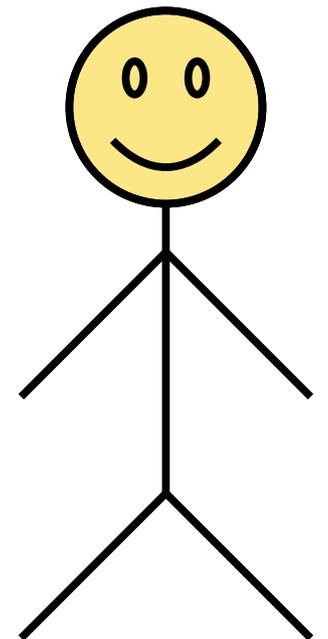
***Proof Writer (You)***

$k = 68$

***Neither Picks***

$n = 137$

***Reader Picks***



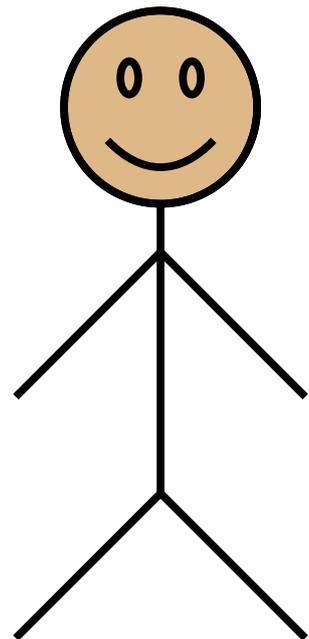
***Proof Reader***

# Proofs as a Dialog

Let  $n$  be an arbitrary odd integer.

Since  $n$  is an odd integer, there is an integer  $k$  such that  $n = 2k + 1$ .

Now, let  $z = k - 34$ .



***Proof Writer (You)***

$$z = 34$$

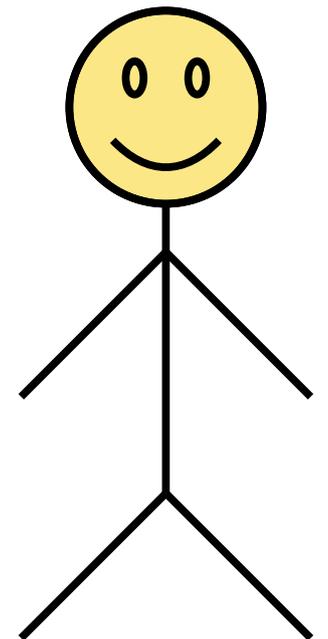
***Writer Picks***

$$k = 68$$

***Neither Picks***

$$n = 137$$

***Reader Picks***



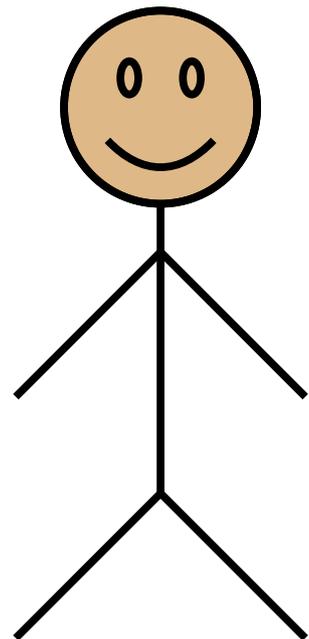
***Proof Reader***

# Proofs as a Dialog

Let  $n$  be an arbitrary odd integer.

Since  $n$  is an odd integer, there is an integer  $k$  such that  $n = 2k + 1$ .

Now, let  $z = k - 34$ .



*Proof Writer (You)*

$z = 34$

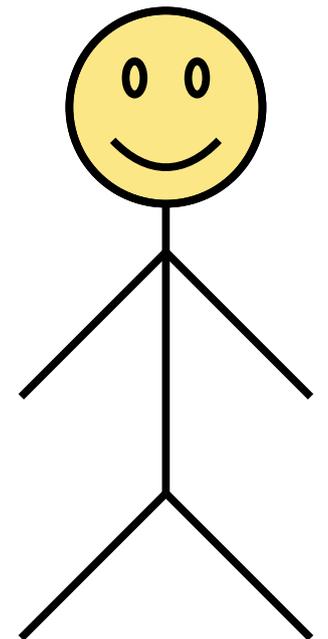
*Writer Picks*

$k = 68$

*Neither Picks*

$n = 137$

*Reader Picks*

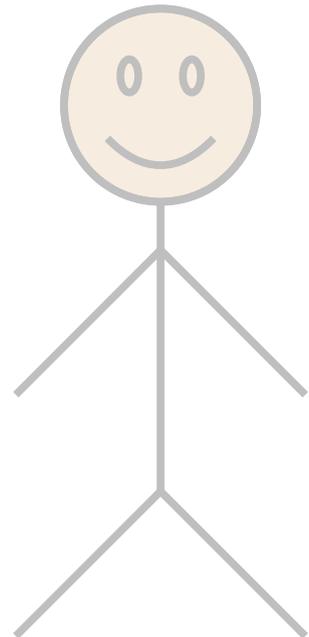


*Proof Reader*

Each of these variables has a distinct, assigned value.

Since Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Now, let  $z = k - 34$ .



*Proof Writer (You)*

$z = 34$

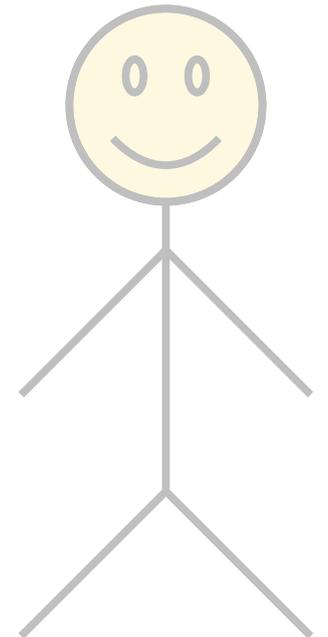
*Writer Picks*

$k = 68$

*Neither Picks*

$n = 137$

*Reader Picks*



*Proof Reader*

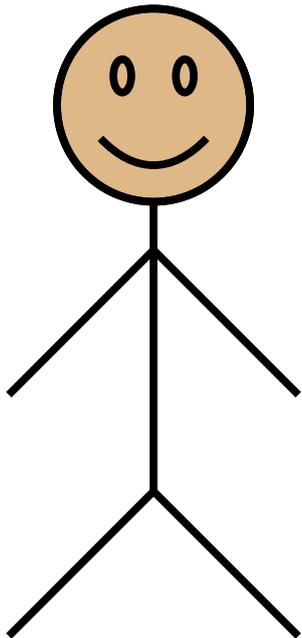
# Who Owns What?

- The **reader** chooses and owns a value if you use wording like this:
  - Pick a natural number  $n$ .
  - Consider some  $n \in \mathbb{N}$ .
  - Fix a natural number  $n$ .
  - Let  $n$  be a natural number.
- The **writer** (you) chooses and owns a value if you use wording like this:
  - Let  $r = n + 1$ .
  - Pick  $s = n$ .
- **Neither** of you chooses a value if you use wording like this:
  - Since  $n$  is even, we know there is some  $k \in \mathbb{Z}$  where  $n = 2k$ .
  - Because  $n$  is odd, there must be some integer  $k$  where  $n = 2k + 1$ .

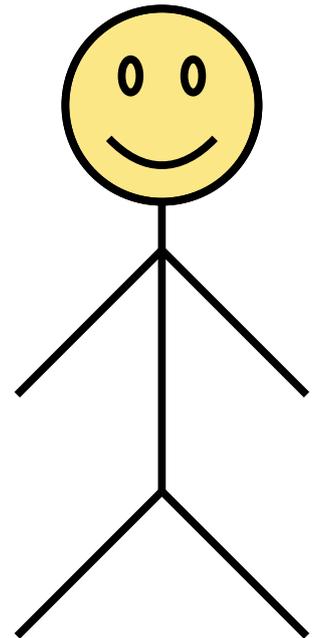
# Proofs as a Dialog

Let  $x$  be an arbitrary even integer.

Then for any even  $x$ , we know that  $x+1$  is odd.



*Proof Writer (You)*

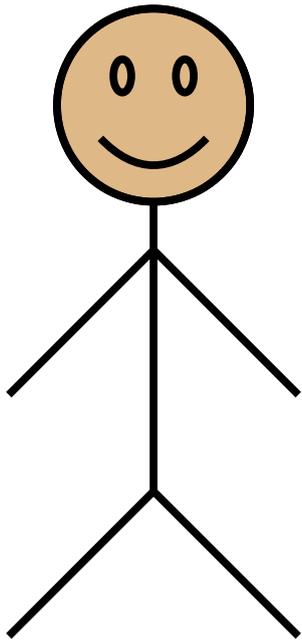


*Proof Reader*

# Proofs as a Dialog

Let  $x$  be an arbitrary even integer.

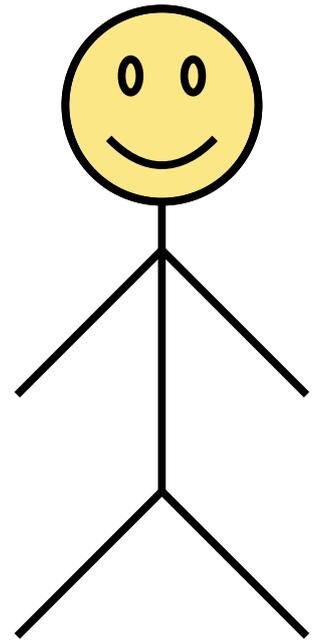
Then for any even  $x$ , we know that  $x+1$  is odd.



***Proof Writer (You)***

$$x = 242$$

***Reader Picks***

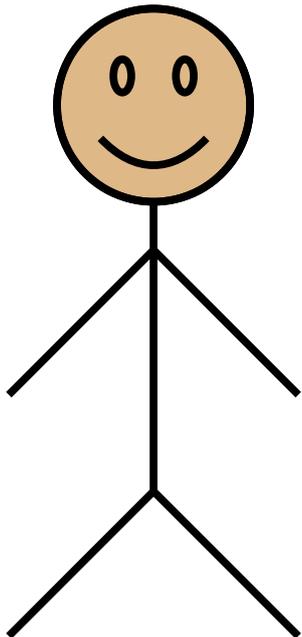


***Proof Reader***

# Proofs as a Dialog

Let  $x$  be an arbitrary even integer.

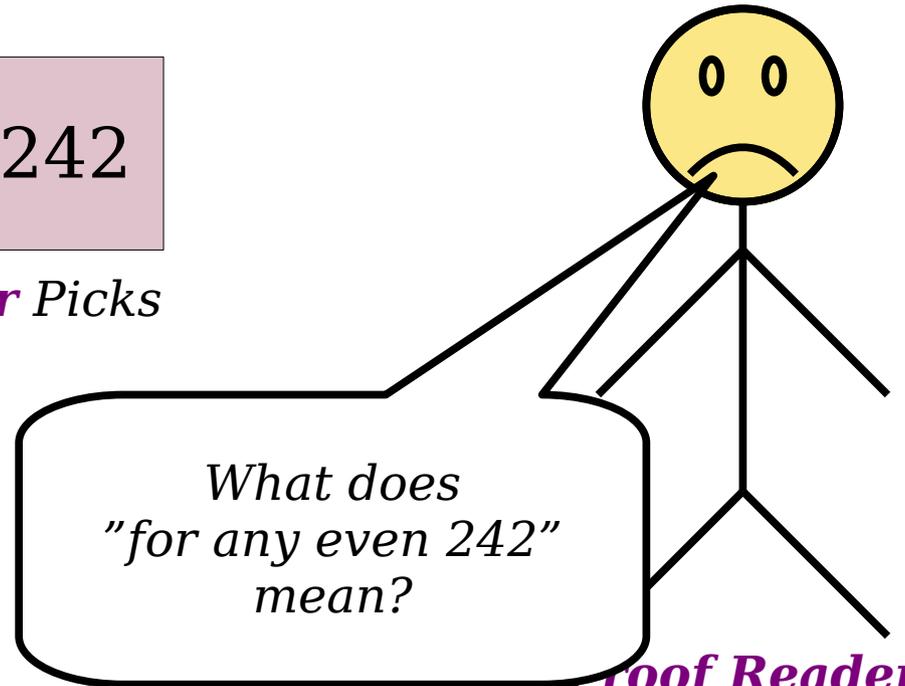
Then **for any even  $x$** , we know that  $x+1$  is odd.



**Proof Writer (You)**

$$x = 242$$

*Reader Picks*

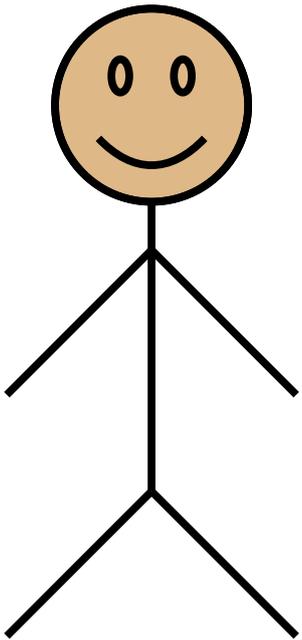


**Proof Reader**

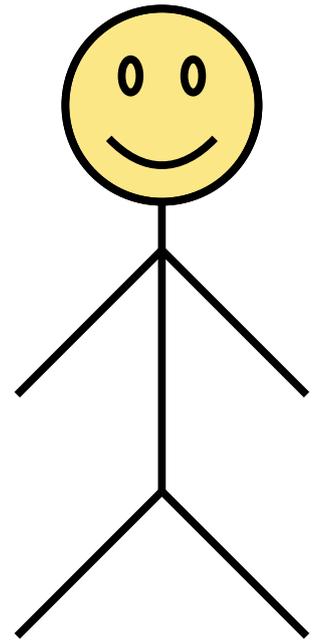
# Proofs as a Dialog

Let  $x$  be an arbitrary even integer.

Since  $x$  is even, we know that  $x+1$  is odd.



*Proof Writer (You)*

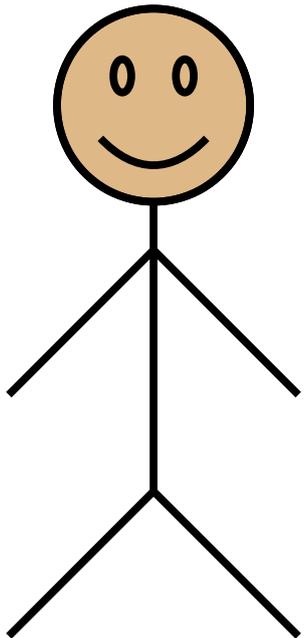


*Proof Reader*

# Proofs as a Dialog

Let  $x$  be an arbitrary even integer.

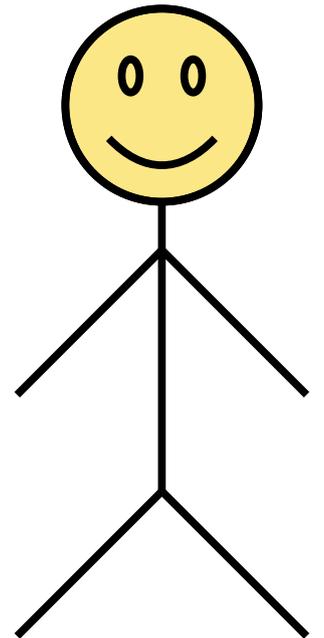
Since  $x$  is even, we know that  $x+1$  is odd.



***Proof Writer (You)***

$$x = 242$$

***Reader Picks***

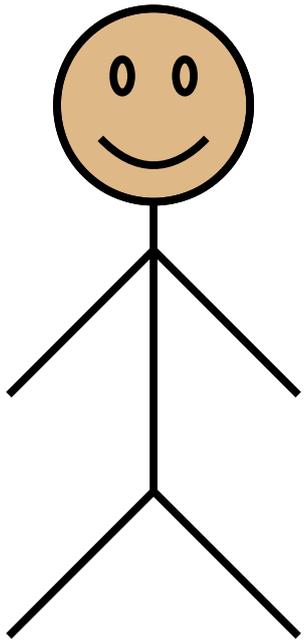


***Proof Reader***

# Proofs as a Dialog

Let  $x$  be an arbitrary even integer.

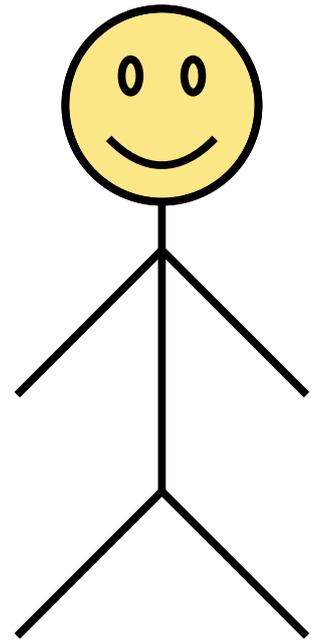
Since  $x$  is even, we know that  $x+1$  is odd.



*Proof Writer (You)*

$$x = 242$$

*Reader Picks*



*Proof Reader*

***Every variable needs a value.***

***Avoid talking about “all  $x$ ” or “every  $x$ ”  
when manipulating something  
concrete.***

***To prove something is true for any  
choice of a value for  $x$ , let the reader  
pick  $x$ .***

***Once you've said something like***

Let  $x$  be an integer.  
Consider an arbitrary  $x \in \mathbb{Z}$ .  
Pick any  $x$ .

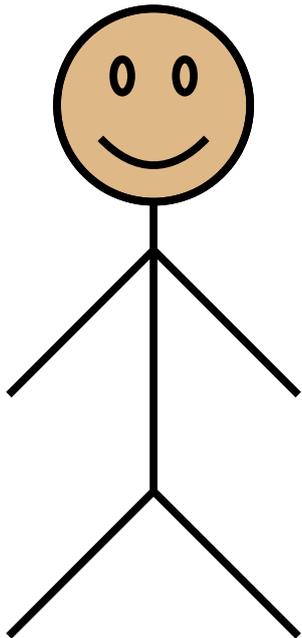
***Do not say things like the following:***

This means that ***for any***  $x \in \mathbb{Z} \dots$   
So ***for all***  $x \in \mathbb{Z} \dots$

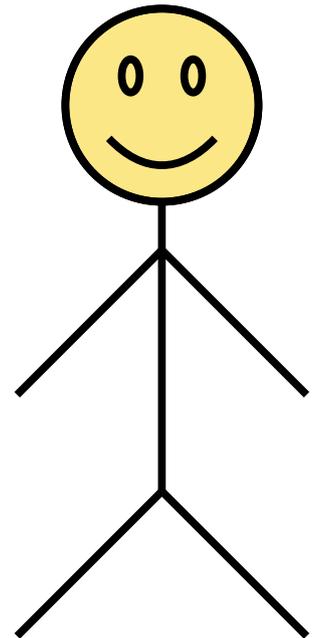
# Proofs as a Dialog

Pick two integers  $m$  and  $n$  where  $m+n$  is odd.

Let  $n = 1$ , which means that  $m+1$  is odd.



***Proof Writer (You)***

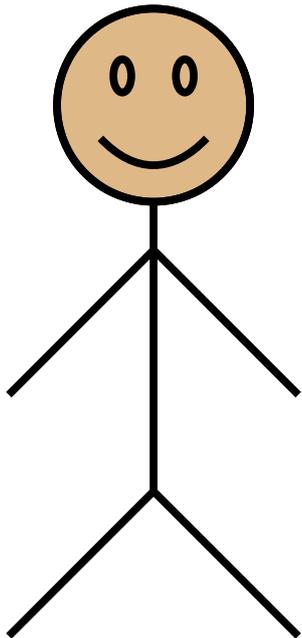


***Proof Reader***

# Proofs as a Dialog

Pick two integers  $m$  and  $n$  where  $m+n$  is odd.

Let  $n = 1$ , which means that  $m+1$  is odd.



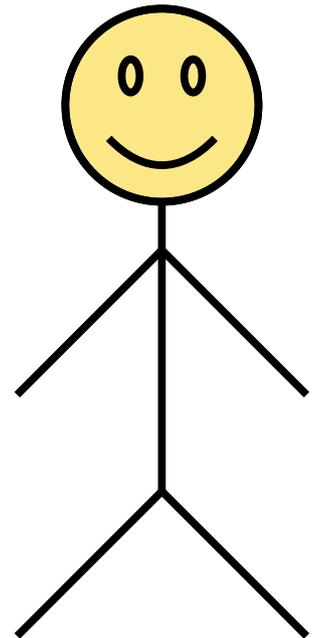
***Proof Writer (You)***

$$m = 103$$

***Reader Picks***

$$n = 166$$

***Reader Picks***

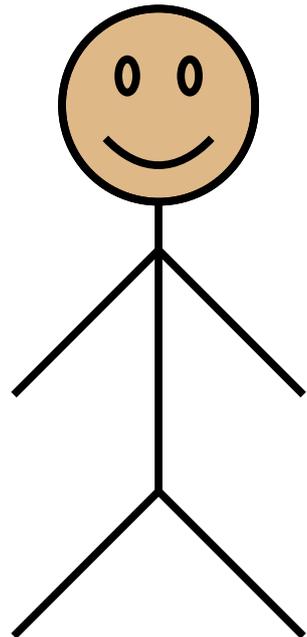


***Proof Reader***

# Proofs as a Dialog

Pick two integers  $m$  and  $n$  where  $m+n$  is odd.

Let  $n = 1$ , which means that  $m+1$  is odd.



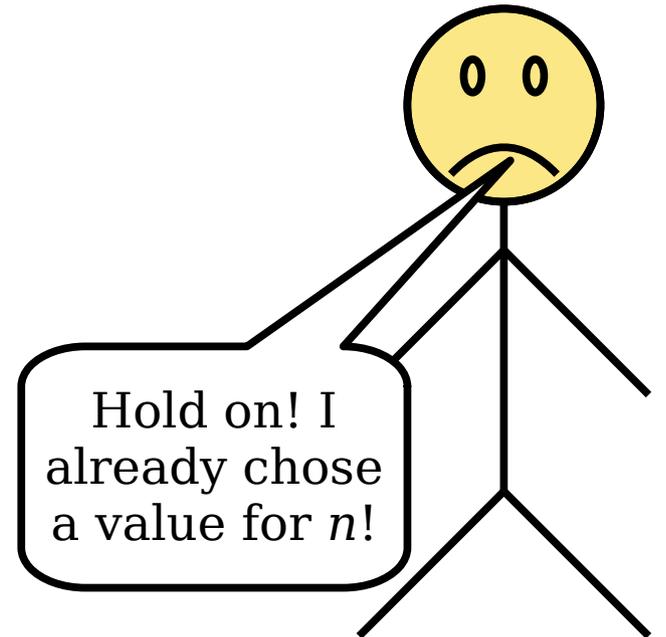
***Proof Writer (You)***

$$m = 103$$

***Reader Picks***

$$n = 166$$

***Reader Picks***

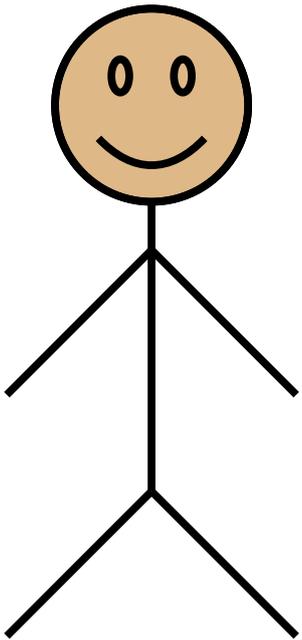


***Proof Reader***

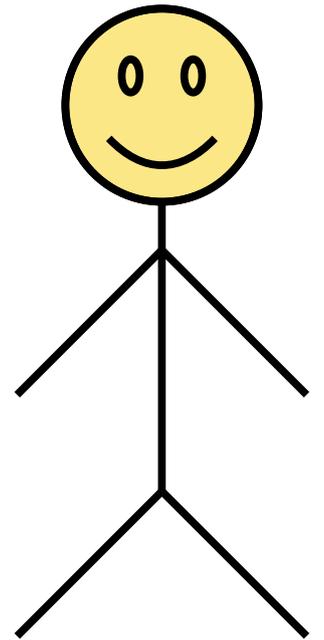
# Proofs as a Dialog

Let  $n = 1$ .

Pick any integer  $m$  where  $m+1$  is odd.



*Proof Writer (You)*

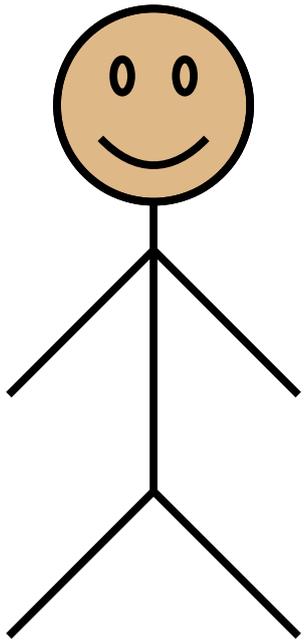


*Proof Reader*

# Proofs as a Dialog

Let  $n = 1$ .

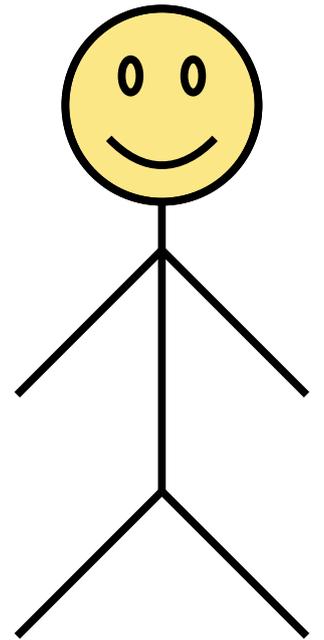
Pick any integer  $m$  where  $m+1$  is odd.



***Proof Writer (You)***

$n = 1$

***Writer Picks***

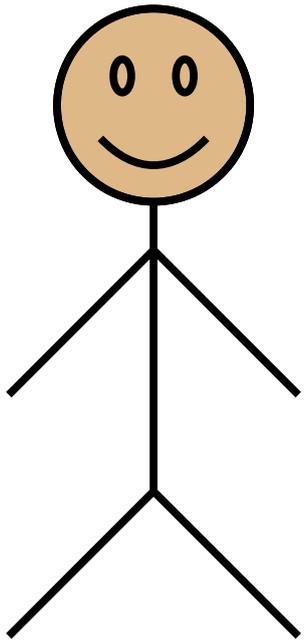


***Proof Reader***

# Proofs as a Dialog

Let  $n = 1$ .

Pick any integer  $m$  where  $m+1$  is odd.



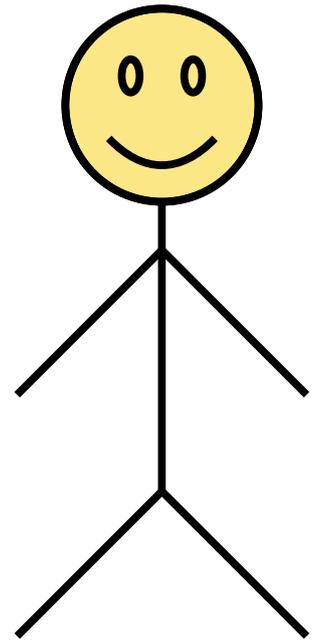
***Proof Writer (You)***

$$m = 166$$

***Reader Picks***

$$n = 1$$

***Writer Picks***



***Proof Reader***

# Proofs as a Dialog

Do we even need  $n$  here?

Let  $n = 1$ .

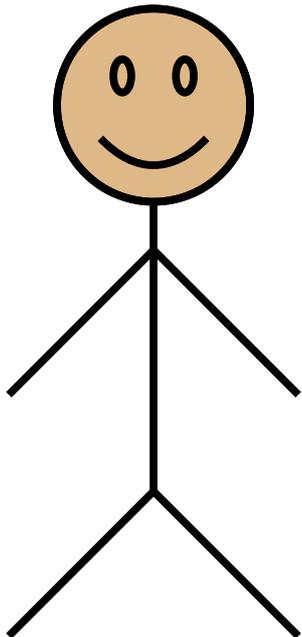
Pick any integer  $m$  where  $m+1$  is odd.

$m = 166$

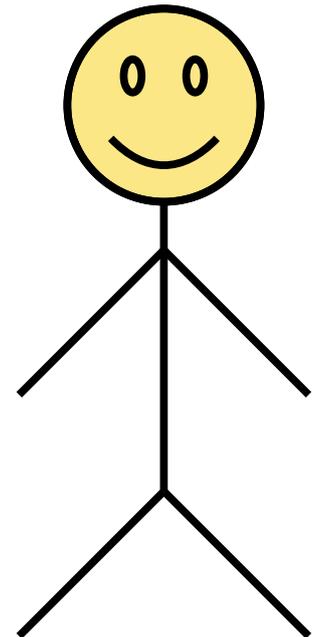
*Reader Picks*

$n = 1$

*Writer Picks*



*Proof Writer (You)*



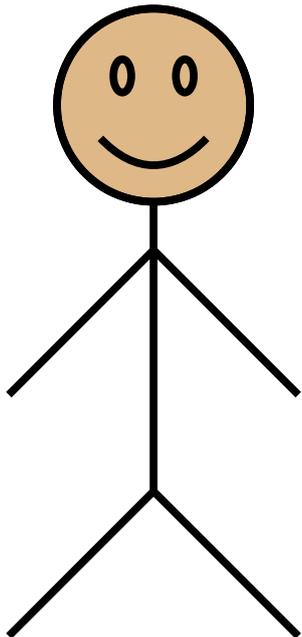
*Proof Reader*

# Proofs as a Dialog

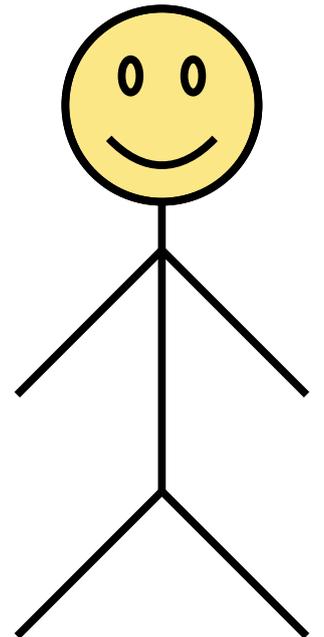
Pick any integer  $m$  where  $m+1$  is odd.

$$m = 166$$

*Reader Picks*



*Proof Writer (You)*



*Proof Reader*

***Be mindful of who owns what variable.***

***Don't change something you don't own.***

***You don't always need to name things,  
especially if they already have a name.***